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Finite Groups All of Whose Non-2-closed 2-Local Subgroups Have Sylow 2-Subgroups of Class 2

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1. INTRODUCTION

Gilman and Gorenstein's classification [3] of finite groups with S_2-subgroups of (nilpotence) class 2 finally deals with the case that all 2-local subgroups are 2-constrained and have trivial cores. The purpose of this paper is to describe an alternate approach to this case. Our method in fact applies to a slightly larger class of groups; that is, finite groups G all of whose 2-local subgroups N that are not 2-closed satisfy the following two conditions:

(i) an S_2-subgroup of N has class at most 2;

(ii) $C_N(O_2(N)) \subseteq O_2(N)$.

We note that the condition (ii) is equivalent to saying that N is 2-constrained and has a trivial core. Hence all 2-local subgroups of G are 2-constrained and have trivial cores unless G is a TI-group. However, as the groups PSL(2, q) with $q = 2^n \pm 1$, $n \geqslant 4$, indicate, G does not necessarily possess an S_2-subgroup of class 2. Indeed, an S_2-subgroup of these groups is dihedral of order 2^n and so has class $n - 1$, while their maximal 2-local subgroup is either a 2-group or isomorphic to the symmetric group of degree 4 and so satisfies both (i) and (ii). We note also that a group N satisfying (ii) can not have Abelian S_2-subgroups unless it is 2-closed; so we may rewrite (i) as follows: an S_2-subgroup of N has class 2.

With these remarks in mind, we now state our result.

THEOREM. *Let G be a nonidentity finite group with $O(G) = 1 = O_2(G)$ in which each 2-local subgroup N which is not 2-closed satiesfies the conditions* (i) *and* (ii). *Then* $\mathrm{Inn}(K) \subseteq G \subseteq \mathrm{Aut}(K)$, *where K is isomorphic to one of the simple groups on the following list*:

(a) $\mathrm{PSL}(2, q)$, $q = 2^n \pm 1 > 5$;

(b) $\mathrm{PSL}(2, 2^n)$, $\mathrm{Sz}(2^n)$, $\mathrm{PSU}(3, 2^n)$, $n \geqslant 2$;

(c) $\mathrm{PSL}(3, 2^n)$, $\mathrm{PSp}(4, 2^n)$, $n \geqslant 2$.

We actually derive our theorem from Theorem A of our previous paper [6], namely Theorem 2.5 presented in Section 2. Therefore, given a group G satisfying the hypothesis of our theorem, our task will be to show that each maximal 2-local subgroup N which is not 2-closed satisfies the conditions (1), (2), (3) of Theorem 2.5. It will be accomplished in Section 4. In fact, we can prove more, so that we need not proceed exactly as in the proof of Theorem 2.5. We can for instance avoid using Goldschmidt's "strong closure" theorem [5] which was necessary for the proof of Theorem 2.5.

Section 4 is very close to a portion of Section 4 of [6], and deals with a pair (D_1, D_2) of distinct maximal S_2-intersections contained in an S_2-subgroup P of G such that $N_P(D_1) = N_P(D_2)$ and either D_i are normal in P or D_1, D_2 are permuted by an element of P. The existence of such a pair (Lemma 4.1) is a consequence of some general consideration in Section 3. Let G be an arbitrary finite group, $\mathscr{H}$ the family of 2-subgroups defined by Suzuki [9, p. 198], and $\mathscr{H}_0$ the subset of $\mathscr{H}$ consisting of all H in $\mathscr{H}$ such that $N_G(H)/H$ has a strongly embedded subgroup. It is known that $\mathscr{H}_0$ controls the intersection of two distinct S_2-subgroups (Proposition 3.1). Using this property of $\mathscr{H}_0$, we prove the following key result:

If $O(G) = 1 = O_2(G)$, then either $\mathscr{H}_0$ is empty or each S_2-subgroup of G contains at least two elements of $\mathscr{H}_0$.

For the proof of this fact, we also require a technique of Suzuki [10, (2.19)], which enables one to find a strongly embedded subgroup in a group. This technique is again used to show that $\Omega_1(Z(Q_1)) \not\subseteq D_2$ for each S_2-subgroup $Q_1 \neq N_P(D_1)$ of $N_G(D_1)$ (Lemma 4.7). Once this is established, it readily follows that G satisfies the hypothesis of Theorem 2.5.

In general, we follow the terminology and notation of [7]. Besides we use the following:

S_2-intersection $=$ an intersection of two distinct S_2-subgroups;

$\mathscr{S}(G) =$ the set of S_2-subgroups of G;

$N(H) =$ the normalizer of H in G, in case there is no danger of confusion;

$C(H) =$ the centralizer of H in G, in case there is no danger of confusion;

$G^2 =$ the subgroup of G generated by the squares of elements of G;

$O'(G) =$ the subgroup of G generated by the 2-elements of G.

All groups are assumed to be finite from now on.

2. Assumed Results

In this section, G will denote an arbitrary finite group. We collect here all of the assumed results which we shall need for the proof of our theorem.

Theorem 2.1 (Brauer and Suzuki [2], Burnside [7, Theorem 7.4.3]). *Suppose that an S_2-subgroup of G is either cyclic or generalized quaternion. Then $G/O(G)$ has exactly one involution.*

Theorem 2.2 (Suzuki [8, Theorem 2, 3, 6]). *Suppose that G is a* TI-*group which is not 2-closed. Then one of the following statements holds:*

(1) *G is solvable and an S_2-subgroup of G is either cyclic or generalized quaternion;*

(2) *$O'(G)/Z(O'(G))$ is isomorphic to one of the following groups:* $\mathrm{PSL}(2, 2^n)$, $\mathrm{Sz}(2^n)$ *and* $\mathrm{PSU}(3, 2^n)$, $n \geqslant 2$.

Theorem 2.3 (Bender [1]). *Suppose that G has a strongly embedded subgroup. Then one of the following statements holds:*

(1) *an S_2-subgroup of G is either cyclic or generalized quaternion;*

(2) *$O'(G)\,O(G)/O(G)$ is isomorphic to one of the following groups:* $\mathrm{PSL}(2, 2^n)$, $\mathrm{Sz}(2^n)$ *and* $\mathrm{PSU}(3, 2^n)$, $n \geqslant 2$.

Theorem 2.4 (Goldschmidt [4]). *Suppose that an S_2-subgroup P of G has class n, $n > 1$. Then $\mho^{n-1}(Z(P)) \subseteq O_{2',2}(G)$.*

Theorem 2.5 (Gomi [6]). *Suppose that each maximal 2-local subgroup N of G which is not 2-closed satisfies the following conditions:*

(1) $C_N(O_2(N)) \subseteq O_2(N)$;

(2) *$N/O_2(N)$ is a* TI-*group;*

(3) *each involution of $O_2(N)$ lies in the center of some S_2-subgroup of N. If furthermore $O(G) = O_2(G) = 1 \neq G$, then* $\mathrm{Inn}(K) \subseteq G \subseteq \mathrm{Aut}(K)$ *where K is isomorphic to one of the following groups:*

(a) $\mathrm{PSL}(2, q)$ *with* $q = 2^n \pm 1 > 5$, $\mathrm{PSL}(3, 3)$, M_{11};

(b) $\mathrm{PSL}(2, 2^n)$, $\mathrm{Sz}(2^n)$, $\mathrm{PSU}(3, 2^n)$, $n \geqslant 2$;

(c) $\mathrm{PSL}(3, 2^n)$, $\mathrm{PSp}(4, 2^n)$, $n \geqslant 2$.

3. Preliminary Results

Let G denote an arbitrary group. As in [6], we define $\mathscr{H}$ to be the family of 2-subgroups H of G satisfying the following conditions:

(1) $H \neq 1$;

(2) $N(H)$ is not 2-closed;

(3) $H = O_2(N(H))$.

Furthermore, we let $\mathscr{H}_0$ denote the subset of $\mathscr{H}$ consisting of all H in $\mathscr{H}$ such that $N(H)/H$ has a strongly embedded subgroup. If $P \in \mathscr{S}(G)$, $\mathscr{H}(P)$ (resp. $\mathscr{H}_0(P)$) will denote the set of the elements of $\mathscr{H}$ (resp. $\mathscr{H}_0$) contained in P. We shall regard $\mathscr{H}(P)$ as a partially ordered set with respect to the usual order defined by inclusion. Let $\mathscr{D}(P)$ denote the set of maximal elements of $\mathscr{H}(P)$. In other words, $\mathscr{D}(P)$ is the set of maximal S_2-intersections $\neq 1$ of G contained in P. Hence $\mathscr{D}(P) \subseteq \mathscr{H}_0(P)$, and $N_P(D) \in \mathscr{S}(N(D))$ for each $D \in \mathscr{D}(P)$.

The importance of the family $\mathscr{H}_0$ is the following:

Proposition 3.1 [6, (2.3)]. *Let P, Q be distinct S_2-subgroups of G such that $P \cap Q \neq 1$. Then there exists a sequence $P_0 = P$, $P_1, \ldots, P_n = Q$ of S_2-subgroups of G which satisfies the following conditions*:

(1) $H_i = P_{i-1} \cap P_i$ *is a tame intersection of* P_{i-1} *and* P_i;

(2) $H_i \in \mathscr{H}_0$;

(3) $P \cap Q = H_1 \cap \cdots \cap H_n$.

Using this, we prove

Proposition 3.2. *Let $P \in \mathscr{S}(G)$. If $O(G) = 1 = O_2(G)$, then either $\mathscr{H}_0(P)$ is empty (in which case G is a* TI-*group) or $\mathscr{H}_0(P)$ has no unique minimal element.*

Proof. Suppose that $\mathscr{H}_0(P)$ has a unique minimal element $H(P)$. We first show that if $Q \in \mathscr{S}(G)$ and $P \cap Q \neq 1$, then $H(P) = H(Q)$. In view of Proposition 3.1, we may assume that $P \cap Q$ contains an element H of $\mathscr{H}_0$. It then follows that $H(P) \subseteq H \subseteq Q$ and consequently $H(Q) \subseteq H(P)$. As $|H(P)| = |H(Q)|$, we conclude that $H(P) = H(Q)$, as asserted. We next argue that $N = N(H(P))$ is a strongly embedded subgroup of G. It is clear that N is a proper subgroup of G of odd index. Suppose $x \in G$ and $|N \cap N^x|$ is even. Then there exist Q, R in $\mathscr{S}(N)$ such that $Q \cap R^x \neq 1$. As shown above, $H(Q) = H(R^x) = H(R)^x$. By the same reason, $H(Q) = H(P) = H(R)$ because $H(P) \subseteq Q \cap R$. Hence $x \in N$. Therefore N is strongly embedded in G. Since $O(G) = 1 = O_2(G)$, the Theorems 2.1 and 2.3 immediately imply that G is a TI-group and therefore $\mathscr{H}_0(P)$ is empty. This contradiction completes the proof of Proposition 3.2.

As an immediate consequence, we have the following.

COROLLARY 3.3. *Let $P \in \mathscr{S}(G)$. If $O(G) = 1 = O_2(G)$, then either $\mathscr{H}_0(P)$ is empty or $|\mathscr{H}_0(P)| \geqslant 2$.*

We conclude with the following result of a different nature:

PROPOSITION 3.4 [6, (4.41)]. *Let N be a group with $C_N(O_2(N)) \subseteq O_2(N)$ and with S_2-subgroups of class 2. If P, Q are distinct S_2-subgroups of N, then $Z(P) \neq Z(Q)$.*

4. PROOF OF THE THEOREM

We let G be a nonidentity group with $O(G) = 1 = O_2(G)$ in which each 2-local subgroup N which is not 2-closed satisfies the conditions (i) and (ii). As was remarked in Section 1, our aim is to prove that G satisfies the hypothesis of Theorem 2.5.

We first note that each $H \in \mathscr{H}$ has the following useful properties by virtue of the conditions (i) and (ii):

(a) $C(H) \subseteq H$;

(b) an S_2-subgroup of $N(H)$ has class 2;

(c) if R is a 2-subgroup of G of class 2 containing H, then $R \subseteq N(H)$.

These facts are used implicitly throughout the paper.

We first prove the following.

LEMMA 4.1. *Suppose $P \in \mathscr{S}(G)$ and $D_1 \in \mathscr{D}(P)$. Then there exists $D_2 \in \mathscr{D}(P)$ which satisfies the following conditions:*

(1) $D_1 \neq D_2$;

(2) $N_P(D_1) = N_P(D_2)$;

(3) *either $D_i \lhd P$ or there exists $u \in P$ such that $D_1{}^u = D_2$ and $D_2{}^u = D_1$.*

Proof. If $D_1 \ntriangleleft P$, take $u \in N_P(N_P(D_1)) - N_P(D_1)$ such that $u^2 \in N_P(D_1)$. Then $D_2 = D_1{}^u$ satisfies all the requirements. Suppose therefore that $D_1 \lhd P$, in which case P has class 2 and so every element of $\mathscr{H}(P)$ is normal in P. It will thus suffice to show that $|\mathscr{D}(P)| \geqslant 2$. Suppose, by way of contradiction, that $\mathscr{D}(P) = \{D\}$. In view of Corollary 3.3, let $D \neq H \in \mathscr{H}_0(P)$ and set

$N = N(H)$. Since P has class 2, Theorem 2.4 implies that $Z(P)$ is elementary, and so is $P/Z(P)$. Accordingly, P/H is elementary Abelian of order at least 4 because $H \subset D \subset P$. It now follows from Theorem 2.3 that $\bar{N} = N/O_{2,2'}(N)$ has a normal subgroup of odd index isomorphic to $\mathrm{PSL}(2, 2^n)$, $n \geqslant 2$. As a consequence, elements of $\bar{P}^{\#}$ are conjugate to each other in $N_{\bar{N}}(\bar{P})$. However on the other hand, $1 \subset \bar{D} \subset \bar{P}$ and $\bar{D} \lhd N_{\bar{N}}(\bar{P})$ because $\mathscr{D}(P) = \{D\}$. This contradiction completes the proof of Lemma 4.1.

DEFINITION. If $P \in \mathscr{S}(G)$, $\mathscr{E}(P)$ will denote the set of the pairs (D_1, D_2) of elements of $\mathscr{D}(P)$ which satisfy the conditions (1), (2), (3) of Lemma 4.1.

Remaining lemmas involve the following situation:

HYPOTHESIS 4.2.

(1) $P \in \mathscr{S}(G)$;

(2) $(D_1, D_2) \in \mathscr{E}(P)$;

(3) $N_P(D_i) \neq Q_i \in \mathscr{S}(N(D_i))$;

(4) $Q_i \subseteq P_i \in \mathscr{S}(G)$.

Furthermore, we let

$$N_i = N(D_i), \quad Q = N_P(D_i), \quad X = \Omega_1(Z(Q)) \quad \text{and} \quad X_i = \Omega_1(Z(Q_i)).$$

LEMMA 4.3. *Under Hypothesis* 4.2, $X_1 \neq X \neq X_2$.

Proof. By symmetry, we need only prove that $X_1 \neq X$. Suppose $X_1 = X$. Then $N(X)$ is not 2-closed because $Q, Q_1 \subseteq N(X)$, whence $N_P(X)$ has class 2 by (i). Hence $N_P(X) = Q$ and so $P = Q$ because $N_P(Q) \subseteq N_P(X)$. But then by Theorem 2.4, $Z(P) = X = Z(P_1)$, contradicting Proposition 3.4. Hence Lemma 4.3 holds.

LEMMA 4.4. *Under Hypothesis* 4.2, *if* $x \in X - X_i$, $i = 1$ *or* 2, *then the following conditions hold*:

(1) $C_{P_i}(x) = D_i$;

(2) $Q/D_i \cong [Q_i, x] \subseteq X_i$;

(3) $[Q_i, x] \cap X = 1$.

Proof. By symmetry, we need only consider the case $i = 1$. Suppose that $D_1 \subset C_{P_1}(x)$. Then as D_1 is a maximal S_2-intersection, $C_{P_1}(x) \in \mathscr{S}(C(x))$. Moreover, $C(x)$ is not 2-closed and so an S_2-subgroup of $C(x)$ has class 2 by (i). Consequently, $C_P(x) = Q$ and $C_{P_1}(x) = C_{Q_1}(x)$, which obviously is a contradiction because $x \notin Z(Q_1)$. Hence (1) holds.

Since Q_1 has class 2, the mapping $y \to [y, x]$ defines a homomorphism of Q_1 onto $[Q_1, x]$. Likewise $[Q_1, x] \subseteq X_1$ because $x^2 = 1$. By (1), the kernel of this homomorphism is equal to D_1, so $Q_1/D_1 \cong [Q_1, x]$. Hence (2) holds.

Suppose that $[Q_1, x] \cap X \neq 1$. Then there exists a in $Q_1 - D_1$ such that $[a, x] \in X$. Set $A = \langle [a, x], x \rangle$ and let $b \in Q - D_1$. Then A is a four-group and is normalized by ab. Indeed, $[a, x] \in X_1 \cap X$ is centralized by ab, and ab permutes x and $[a, x]x$. However, $\langle ab \rangle / C_{\langle ab \rangle}(A)$ is a homomorphic image of $\langle ab \rangle D_1/D_1$ and so has odd order because N_1/D_1 is a TI-group. This contradiction shows that (3) also holds.

LEMMA 4.5. *Under Hypothesis* 4.2, *the following conditions hold*:

(1) *if* $X_2 \not\subseteq D_1$, *then* $X_1 \cap D_2 = X_1 \cap X$;

(2) *if* $X_1 \not\subseteq D_2$, *then* $X_2 \cap D_1 = X_2 \cap X$;

(3) $X_1 \subseteq D_2$ *if and only if* $X_2 \subseteq D_1$.

Proof. Suppose $X_2 \not\subseteq D_1$ and $x \in X_1 \cap D_2$. If N_1 is solvable, then $|Q : D_1| = 2$ by Theorem 2.2 and Lemma 4.4, (2), whence

$$x \in \Omega_1(C_Q(\langle D_1, X_2 \rangle)) = X.$$

Hence (1) holds in this case. Assume therefore that N_1 is non-solvable. We have $\langle Q_1, X_2 \rangle \subseteq C_{O'(N_1)}(x)$ and hence $C_{O'(N_1)}(x)/D_1$ is not 2-closed. Since $O'(N_1)/D_1$ is a central extension of PSL(2, 2^n), $n \geqslant 2$, by Theorem 2.2, it follows that $C_{O'(N_1)}(x) = O'(N_1)$ and therefore $x \in X$. Hence (1) holds in this case as well. By Symmetry, (2) also holds. (3) is immediate from (1), (2) and Lemma 4.3.

DEFINITION. Let $P \in \mathscr{S}(G)$ and let $(D_1, D_2) \in \mathscr{E}(P)$. Then $(D_1, D_2) \in \mathscr{E}'(P)$ if and only if $X_1 \not\subseteq D_2$ and $X_2 \not\subseteq D_1$, where notation is as in Hypothesis 4.2. This definition is independent of the choice of Q_i by Lemma 4.5, (3).

LEMMA 4.6. *Under Hypothesis* 4.2, *if* $(D_1, D_2) \in \mathscr{E}'(P)$, *then the following conditions hold*:

(1) $D_i^2 = 1$;

(2) $D_i = X_i X$;

(3) *each involution of* D_i *lies in the center of some* S_2*-subgroup of* N_i;

(4) *every elementary Abelian subgroup of* Q *is contained in either* D_1 *or* D_2.

Proof. Let $x \in X - X_1$. Then by Lemma 4.4,

$$|Q : D_1| = |[Q_1, x]| \leqslant |X_1 : X_1 \cap X|.$$

Moreover by the preceding lemma,

$$|X_1 : X_1 \cap X| = |X_1 : X_1 \cap D_2| \leqslant |Q : D_2|.$$

By virtue of the symmetry between D_1 and D_2, we conclude that $Q = X_1D_2 = X_2D_1$ and $|X: X_i \cap X| = |Q : D_i|$. Setting $D = D_1 \cap D_2$, we have as a consequence that $D_i = X_iD$ and then $D_i^2 = D^2$. Accordingly, $Q_i \subseteq N_{P_i}(D^2) \in \mathscr{S}(N(D^2))$ because D_i is a maximal S_2-intersection, and so $N(D^2)$ is not 2-closed. Thus if $D^2 \neq 1$, then $N_{P_i}(D^2)$ has class 2 by (i), whence $N_{P_i}(D^2) = Q_i \in \mathscr{S}(N(D^2))$ and so $X_1 \subseteq O_2(N(D^2)) \subseteq Q_2$. This, however, contradicts our assumption that $(D_1, D_2) \in \mathscr{E}'(P)$. Therefore we must have $D^2 = 1$, and thus (1) holds. Since $Q = D_1D_2$, it immediately follows from this that $D = X$, which proves (2).

To prove (3), we suppose for definiteness that $i = 1$. In case N_1 is solvable, Theorem 2.2 and Lemma 4.4, (2) yield that $|Q : D_1| = 2$ and therefore $D_1/X_1 \cap X$ is a four-group. Hence if $R \in \mathscr{S}(C_{N_1}(X_1 \cap X))$ and $Q \neq R \neq Q_1$, then $D_1 = X \cup X_1 \cup Z(R)$ by Lemma 4.3, which proves (3) in this case. Assume therefore that N_1 is nonsolvable, in which case $O'(N_1)/D_1$ is a central extension of PSL(2, 2^n), $n \geqslant 2$, by Theorem 2.2. As a consequence, the group $X_1 \cap X$ is independent of the choice of Q_1. Likewise N_1 has precisely $|Q : D_1| + 1$ S_2-subgroups, so we have

$$\begin{aligned}
\Big|\bigcup_{R\in\mathscr{S}(N_1)} Z(R)\Big| &= (|Q: D_1| + 1)(|X| - |X_1 \cap X|) + |X_1 \cap X| \\
&= (|X: X_1 \cap X| + 1)(|X| - |X_1 \cap X|) + |X_1 \cap X| \\
&= |X: X_1 \cap X| \cdot |X| \\
&= |D_1|.
\end{aligned}$$

Hence (3) holds in this case as well.

To prove (4), we note that $C_Q(d_i) = D_i$ for each d_i in $D_i - X$, by (2) and Lemma 4.4, (1). Let x be an involution of Q. Since $Q = D_1D_2$, x can be written as $x = d_1d_2$ with $d_i \in D_i$. Thus $1 = x^2 = [d_1, d_2]$ because D_i is elementary Abelian. Since $D_1 \cap D_2 = X$, we have that either $d_1 \in X$ or $d_2 \in X$ and therefore $x \in D_1 \cup D_2$. Hence we conclude that every involution of Q lies in $D_1 \cup D_2$. Consequently (4) holds. The proof of Lemma 4.6 is complete.

LEMMA 4.7. *Under Hypothesis* 4.2, $(D_1, D_2) \in \mathscr{E}'(P)$.

Proof. Suppose that $(D_1, D_2) \notin \mathscr{E}'(P)$. We shall derive a contradiction by showing that G has a strongly embedded subgroup.

(a) *We have* $P = Q$ *and* P *has class* 2.

Proof. Set $D = \Omega_1(D_1 \cap D_2)$. Then $[Q_i, D] \subseteq X_i \subseteq D$, whence $Q_i \subseteq N(D)$. It therefore follows from (i) that $N_P(D)$ has class 2 and so $N_P(D) = Q$. If $P \neq Q$, then by the definition of $\mathscr{E}(P)$, there is $u \in P$ such that $D_1^u = D_2$ and $D_2^u = D_1$, and consequently $u \in N_P(D) - Q$. Therefore we must have $P = Q$.

(b) *$\mathscr{E}'(P)$ is empty.*

Proof. Suppose that $(E_1, E_2) \in \mathscr{E}'(P)$. We have $E_i \lhd P$ by (a), so every elementary Abelian subgroup of P is contained in either E_1 or E_2 by Lemma 4.6, (4). Assume therefore, say, $\langle X_1, X_2 \rangle \subseteq E_1$. We can assume that $E_1 \neq D_1$ and hence that $(D_1, E_1) \in \mathscr{E}(P)$. However, if $P \neq R \in \mathscr{S}(N(E_1))$, then $Z(R) \subseteq D_1$ by Lemma 4.5, (3), and hence $E_1 = Z(R)\,Z(P) \subseteq D_1$ by Lemma 4.6, (2), a contradiction. Hence (b) holds.

(c) *Let $D(P)$ denote the intersection of all elements of $\mathscr{D}(P)$, then $D(P) \in \mathscr{H}$.*

Proof. Let $\mathscr{D}(P) = \{D_i \mid 1 \leqslant i \leqslant n\}$, let $P \neq P_i \in \mathscr{S}(N(D_i))$ and set $D = D(P)$. Since $Z(P) \subseteq D_i$, $D \neq 1$. It follows from (a) that if D_i, $D_j \in \mathscr{D}(P)$ and $D_i \neq D_j$, then $(D_i, D_j) \in \mathscr{E}(P)$ and therefore $Z(P_i) \subseteq D_j$ by (b). Thus $Z(P_i) \subseteq D \subseteq P_i$ and so $P_i \subseteq N(D)$ for each i. Therefore $N(D)$ is not 2-closed and

$$D = \bigcap_{i=1}^{n} D_i = \bigcap_{i=1}^{n} (P \cap P_i) \supseteq O_2(N(D)).$$

This proves (c).

(d) *If $H \in \mathscr{H}_0(P)$, then H is the intersection of elements of $\mathscr{D}(P)$.*

Proof. This is proved in the same way as Lemma 4.1. Let D denote the intersection of all elements of $\mathscr{D}(P)$ that contain H. Then clearly $H \subseteq D \subset P$ and $D \lhd N_{N(H)}(P)$. By the Theorems 2.1, 2.3, and 2.4, either $|P : H| = 2$, or, for $\bar{N} = N(H)/O_{2,2'}(N(H))$, elements of $\bar{P}^{\#}$ are conjugate to each other in $N_{\bar{N}}(\bar{P})$. Hence $H = D$.

(e) *$N(D(P))$ is a strongly embedded subgroup of G.*

Proof. This is proved exactly in the same way as Proposition 3.2. It will suffice to show that $D(P) = D(R)$ whenever $R \in \mathscr{S}(G)$ and $P \cap R \neq 1$. In view of Proposition 3.1, we may assume that $P \cap R$ contains an element H of $\mathscr{H}_0$. By (d), $D(P) \subseteq H \subseteq R$. Hence (c) yields that $R \subseteq N(D(P))$. Therefore the assertion immediately follows from the Sylow Theorem.

By (e) and the Theorems 2.1 and 2.3, we conclude that G is a TI-group, a contradiction because $\mathscr{D}(P)$ is not empty by assumption. Hence Lemma 4.7 holds.

We now conclude the proof of our theorem. Let N be an arbitrary maximal

2-local subgroup which is not 2-closed and set $H = O_2(N)$. Then clearly $H \in \mathscr{H}$ and $N = N(H)$. Let $H \subseteq P \in \mathscr{S}(G)$. Then by Lemma 4.1, there is $(D_1, D_2) \in \mathscr{E}(P)$ such that $H \subseteq D_1$. Furthermore $(D_1, D_2) \in \mathscr{E}'(P)$ by the preceding lemma. Hence by Lemma 4.6, (1), D_1 is elementary Abelian and so $H = D_1$. Therefore N/H is a TI-group, and each involution of H is contained in the center of some S_2-subgroup of N by Lemma 4.6, (3). Hence we conclude that G satisfies the hypothesis of Theorem 2.5. The proof of our theorem is therefore completed.

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